

# Forces on Solitons in Finite, Nonlinear, Planar Waveguides

J. I. Ramos and F. R. Villatoro  
Departamento de Lenguajes y Ciencias de la Computación  
E. T. S. Ingenieros Industriales  
Universidad de Málaga  
Plaza El Ejido, s/n  
29013-Málaga  
SPAIN

## Abstract

The forces acting on and the energies of solitons governed by the nonlinear Schrödinger equation in finite, planar waveguides with periodic and with homogeneous Dirichlet, Neumann and Robin boundary conditions are determined by means of a quantum analogy. It is shown that these densities have *S*-shape profiles and increase as the hardness of the boundary conditions is increased.

KEY TERMS: Spatial solitons, nonlinear Schrödinger equation, finite planar waveguides

# 1 Introduction

Soliton propagation has received a great deal of attention in recent years because of its possible applications in distortionless signal transmission in ultra-high speed and long-distance transoceanic telecommunications by optical fibers [1], soliton lasers, all-optical soliton switches [2], nonlinear planar waveguides [3], etc. However, most of the theoretical studies on one-dimensional solitons have been concerned with the initial-value problem, i.e., infinite intervals in space, or with boundary-generated solitons, i.e., semi-infinite intervals, and have been based on the inverse scattering transform [4], perturbation methods or the infinite sine Fourier transform [5].

Since nonlinear transmission lines, optical fibers, etc., have a finite length, the infinite and semi-infinite spatial domains used in most previous studies of nonlinear wave propagation are idealizations. In this letter, the nonlinear Schrödinger equation in finite intervals with periodic and homogeneous Dirichlet, Neumann and Robin boundary conditions is studied numerically in order to determine the energy of and the forces on solitons prior to, during and after their interaction with the boundaries. The energy of and the forces on solitons are evaluated by means of a quantum analogy between wave propagation phenomena governed by the nonlinear Schrödinger equation and the motion of a particle in a nonlinear potential well. Bian and Chan [3] have previously studied the nonlinear and diffraction forces on solitons governed by the initial-value problem of the nonlinear Schrödinger equation. These authors, however, used a filter and their force and energy densities are not consistent with a strict quantum mechanics analogy.

## 2 One-Dimensional, Nonlinear Schrödinger Equation

The envelope of the electric field in a two-dimensional medium with a quadratic nonlinear refractive index,  $n=n_0+n_2|E'|^2$ , i.e., subject to the Kerr effect, with neither losses nor other higher order effects, can be modelled by the following one-dimensional nonlinear Schrödinger equation [6]

$$2ik\frac{\partial E'}{\partial z'} + \frac{\partial^2 E'}{\partial x'^2} + k^2\frac{n_2^2}{n_0^2}|E'|^2E' = 0 \quad (1)$$

where  $E'$  is the envelope of the electric field,  $k$  is the wave number in a linear medium with  $n=n_0$ ,  $x'$  and  $z'$  are spatial coordinates, and  $n_2$  represents the nonlinear part of the refractive index. Within certain power constraints, the solution of Eq. (1) is a spatial soliton which can propagate in a slab waveguide of width much larger than the transverse mode size of the soliton.

Using the following changes of variables  $E'=n_0/kn_2u$ ,  $z'=2kt$ ,  $x'=x$ , Eq. (1) can be written in the following dimensionless form

$$iu_t = -u_{xx} + Vu, \quad x \in \mathcal{D}, \quad t \geq 0 \quad (2)$$

where  $V=-|u|^2$  and  $\mathcal{D}$  is the spatial domain.

Equation (2) can be written in the standard, dimensional form nonlinear Schrödinger equation by transforming the independent variables as indicated in [3]. In this letter, however, the dimensionless Eq. (2) is used to draw an analogy between the propagation of a soliton in a waveguide and the motion of a particle in a nonlinear potential wave. This analogy allows to determine the dimensionless quantum momentum and quantum energy of

and the dimensionless forces on the soliton as follows. The linear quantum momentum is defined by the one-dimensional operator [7]

$$P \equiv -i \frac{\partial}{\partial x} \quad (3)$$

whose mean value is given by the following time-dependent expression

$$\langle P \rangle = \frac{\langle u, Pu \rangle}{\langle u, u \rangle} = -i \frac{\int_{\mathcal{D}} u^* u_x dx}{\int_{\mathcal{D}} u^* u dx}. \quad (4)$$

where  $u^*$  denotes the complex conjugate of  $u$ .

The quantum momentum density is defined as

$$p(x, t) = -iu^* u_x. \quad (5)$$

The linear quantum energy is defined by the operator

$$E = i \frac{\partial}{\partial t} \quad (6)$$

whose mean value is

$$\langle E \rangle = \frac{\langle u, Eu \rangle}{\langle u, u \rangle} = i \frac{\int_{\mathcal{D}} u^* u_t dx}{\int_{\mathcal{D}} u^* u dx}. \quad (7)$$

while the local energy density is

$$e(x, t) = i \frac{u^* u_t}{\varrho}. \quad (8)$$

where  $\varrho$  is a factor which has been introduced in order to unify our treatment with that of Bian and Chan [3] even though these authors used the dimensional nonlinear Schrödinger equation to analyze the forces on solitons propagating in infinite lines.

A physically and mathematically consistent use of the quantum analogy can be obtained by employing the following renormalization factor

$$\varrho = \int_{\mathcal{D}} u^* u dx \quad (9)$$

which coincides with the first invariant of the initial-value problem of the nonlinear Schrödinger equation. Since depending on the applications of the nonlinear Schrödinger equation and boundary conditions, the first invariant may not remain constant, a value  $\varrho=1$  is used in this work.

The energy density can be split as

$$e(x, t) = -\frac{u^* u_{xx} + |u|^4}{\varrho} = e_k(x, t) + e_v(x, t) \quad (10)$$

where the kinetic and potential energy densities are, respectively,

$$e_k(x, t) = -\frac{u^* u_{xx}}{\varrho}, \quad e_v(x, t) = -\frac{|u|^4}{\varrho}.$$

From the potential energy density, the following nonlinear force density that produces self-focusing on solitons is obtained

$$f_n(x, t) = -\frac{\partial}{\partial x} e_v(x, t) = \frac{\partial}{\partial x} \left( \frac{|u|^4}{\varrho} \right) \quad (11)$$

while, from the kinetic one, the following diffraction force density that is responsible for the diffraction effect on the soliton is obtained

$$f_d(x, t) = -\frac{\partial}{\partial x} e_k(x, t) = \frac{\partial}{\partial x} \left( \frac{u^* u_{xx}}{\varrho} \right). \quad (12)$$

The energy and force densities defined above have complex values. In quantum mechanics, these complex values do not represent any problem because only the mean values of the energy and force can be physically measured

and these values are always real due to the Hermitian quantum operators. In order to obtain physical insight from the energy and force densities defined above, it is convenient to use their real values.

The force densities may be employed to determine their effects on solitons. Bian and Chan [3] used the filter  $\varrho=u^*u$  in order to assess that a soliton can be considered as the result of two opposite phenomena, i.e., the diffraction and the self-focusing caused by dispersion and nonlinearity, respectively, for the initial-value problem of Eq. (2). Note that the force densities obtained by Bian and Chan are somewhat artificial and are not consistent with the quantum mechanics analogy since their renormalization factor is a function of both space and time, and, therefore, affects the values of these densities in a different manner depending on the soliton location at each instant of time.

The numerical results of Bian and Chan [3] for the initial-value problem of the nonlinear Schrödinger equation indicate that the total force density, i.e., the sum of the diffraction and nonlinear force densities, vanishes for the 1-soliton solution. For the general exact  $N$ -soliton solution [8, 9], their definition yields a non-zero total force which is controlled by the nonlinear one indicating that solitons are compressed while propagating.

In this letter, finite planar waveguides in a symmetric, finite interval  $\mathcal{D}=[-L, L]$  subject to the following homogeneous boundary conditions are considered

$$u(-L, t) + \gamma u_x(-L, t) = 0, \quad u(L, t) + \gamma u_x(L, t) = 0, \quad t \geq 0. \quad (13)$$

The values  $\gamma=0$  and  $\infty$  correspond to homogeneous Dirichlet and Neumann boundary conditions, respectively. The nonlinear Schrödinger equation with homogeneous Robin boundary conditions is a skew-symmetric problem in  $x$

except for the limiting cases  $\gamma=0$  and  $\infty$  for which it is symmetric; therefore, the interaction of a soliton with the left boundary is different from that with a right one if  $\gamma$  is finite and different from zero. In this letter, mixed boundary conditions corresponding to  $\gamma=1$  are considered, and the results for the Dirichlet, Neumann and Robin boundary conditions are compared with those for the initial-value problem and with those for the following periodic boundary conditions

$$\frac{\partial^n u}{\partial x^n}(x, t) = \frac{\partial^n u}{\partial x^n}(x+2kL, t), \quad \forall n \geq 0, \quad k \in \mathbb{Z}, \quad x \in \mathcal{D} \equiv [-L, L], \quad t \geq 0. \quad (14)$$

### 3 Presentation of Results

In this section, some sample results (cf. Figs. 1–5) that illustrate the nonlinear force density, the real part of the diffraction force density, the total force density, and the real part of the momentum density are presented as functions of space and time for the four types of boundary conditions considered in this letter. Figures 1–5 also show the space-time isocontours of the three-dimensional data presented in these figures. The results presented in Figs. 1–5 were obtained by means of a Crank-Nicolson finite-difference method and correspond to an interval of  $L=50$ , amplitude and velocity of the soliton equal to one, initial position and initial phase of the soliton equal to zero, and spatial and temporal step sizes equal to 0.25 and 0.01, respectively. These figures show that the nonlinear and diffraction force densities have an *S*-shape, that the nonlinear force density is larger than the diffraction one when the soliton is far away from the boundaries, and that the

momentum density has a bell-shape similar to that of the soliton amplitude.

The boundary does not affect the force and momentum densities on the soliton for the periodic boundary conditions as shown in Fig. 1. However, the force densities are strongly affected by the boundaries for the Robin boundary conditions and the momentum changes sign since the soliton rebounds from the boundary, recovering the shape that it had prior to the collision. Figure 2 corresponds to Dirichlet boundary conditions and shows that the  $S$ -shape of the force densities is maintained during the collision process although very large forces are reached in the collision process since the soliton steepens near the boundary because its amplitude is zero there; the momentum at the boundary is exactly zero and changes sign smoothly.

Figure 3 corresponds to homogeneous Neumann boundary conditions and illustrates the very large values reached by the force densities and the change in their shapes during the collision; the total force density exhibits extrema and the diffraction force exceeds the magnitude of the nonlinear one at the boundary.

Since the nonlinear Schrödinger equation with Robin boundary conditions is not a symmetric problem, the interaction of a soliton with the left boundary is expected to be different from that with the right one. This is clearly illustrated in Figures 4 and 5 which correspond to the first and second collisions, i.e., a collision with the right boundary followed by another one with the left boundary.

Figure 4 indicates that the diffraction force is larger than the nonlinear one only near the boundary resulting in a positive total force. The momentum density behaves similarly to the Neumann case, except that its value



at the right boundary is small but different from zero. Figure 5 shows that the nonlinear (diffraction) force density exhibits a plateau near the boundary during the collision before changing its  $S$ -shape and reaching a negative (positive) extremum at the boundary. Figure 5 indicates that the nonlinear force is always greater than the diffraction one when soliton collides with the left boundary, indicating that the collision with the left boundary is similar to that observed with Dirichlet boundary conditions (cf. Fig. 2). The isocontours presented in Figures 4 and 5 indicate that the soliton becomes closer to the right boundary than to the left one.

## 4 Conclusions

A quantum mechanics analogy is used to determine the forces on and the energies of solitons in finite, nonlinear planar waveguides subject to periodic and to homogeneous Dirichlet, Neumann and Robin boundary conditions in finite intervals. For all the boundary conditions considered in this letter, it has been shown that the nonlinear, diffraction and total force densities have  $S$ -shape profiles and recover the values that they had prior to the interaction with the boundary. The interaction of the soliton with the boundary is characterized by force densities which increase as the hardness of the boundary conditions is increased.

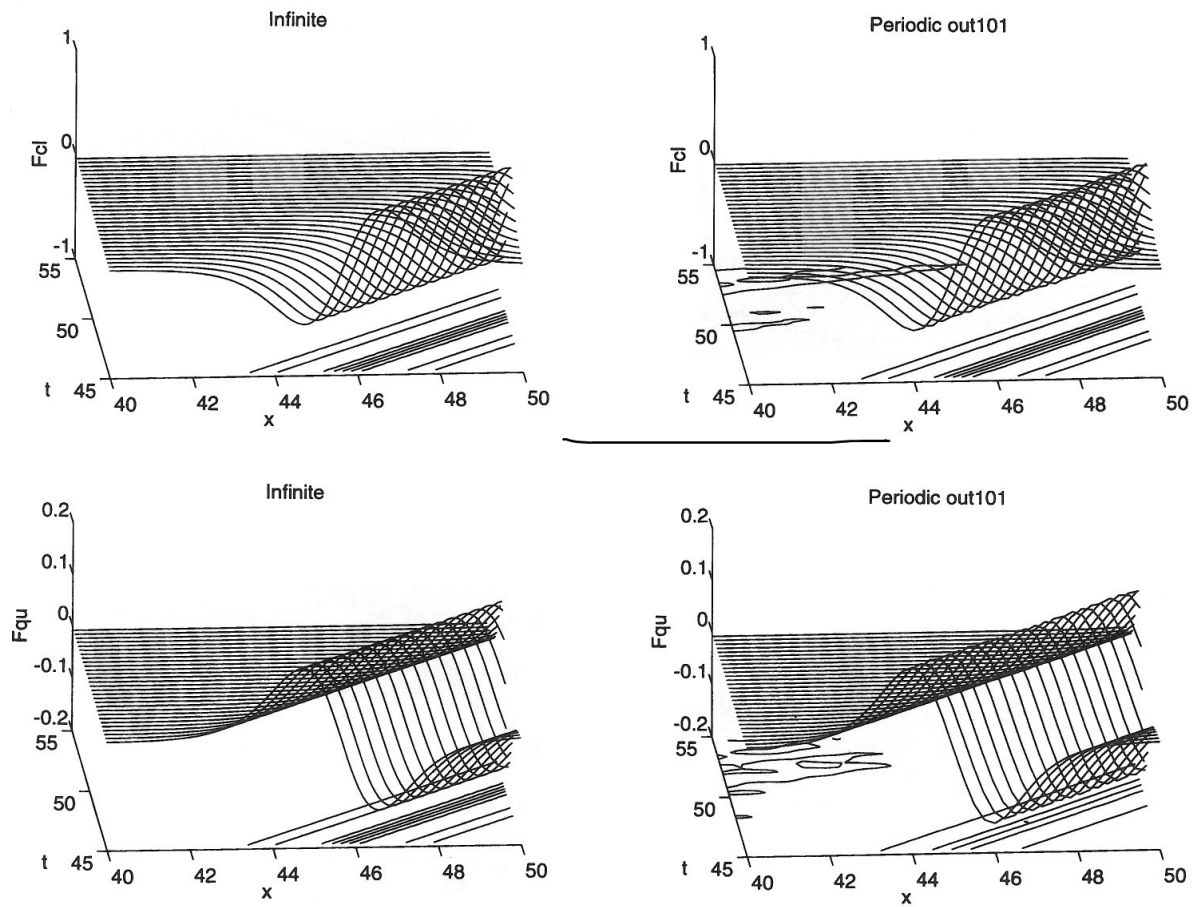
## Acknowledgments

This research was supported by the Spanish D.G.I.C.Y.T. under Project no. PB91-0767. The second author (F.R.V.) has a fellowship from the Programa Sectorial de Formación de Profesorado Universitario y Personal Investigador, Subprograma de Formación de Investigadores "Promoción General del Conocimiento", from the Ministerio de Educación y Ciencia of Spain.

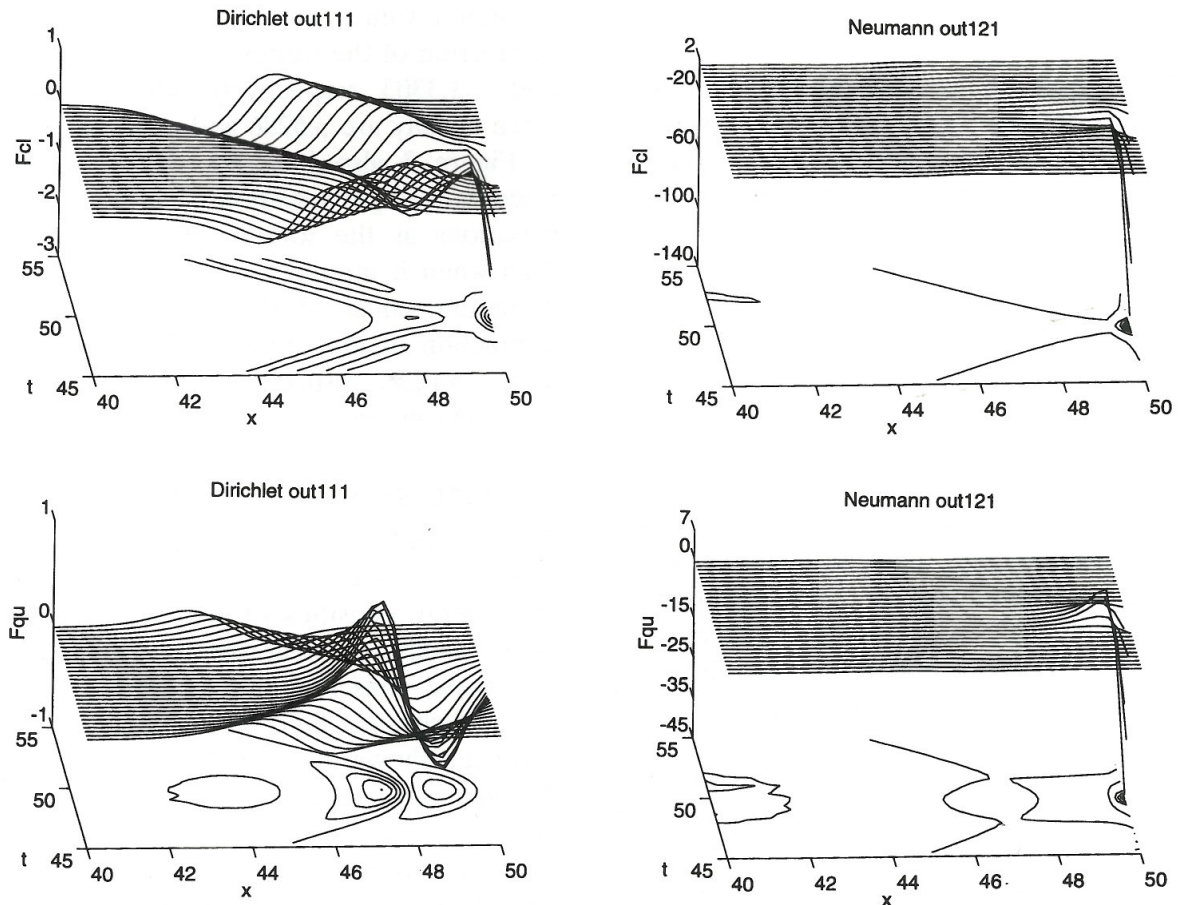
## References

- [1] L. F. Mollenauer and K. Smith, "Demonstration of Soliton Transmission over more than 4000 km in Fiber with Loss periodically compensated by Raman Gain," *Optics Letters*, Vol. 13, 1988, pp. 675-677.
- [2] S. Trillo and S. Wabnitz, "Weak-Pulse-Activated Coherent Soliton Switching in Nonlinear Couplers," *Optics Letters*, Vol. 16, 1991, pp. 1-3.
- [3] J.-R. Bian and A. K. Chan, "Computations of the Diffraction Effect and the Nonlinear Effect on Spatial Solitons in Nonlinear Planar Waveguides," *Microwave and Optical Technology Letters*, Vol. 4, 1991, pp. 184-191.
- [4] M. J. Ablowitz and H. Segur, "*Solitons and the Inverse Scattering Transform*." SIAM, Philadelphia, 1981, p. 313.
- [5] A. S. Fokas, "An Initial-Boundary Value Problem for the Nonlinear Schrödinger Equation," *Physica D*, Vol. 35, 1989, pp. 167-185.

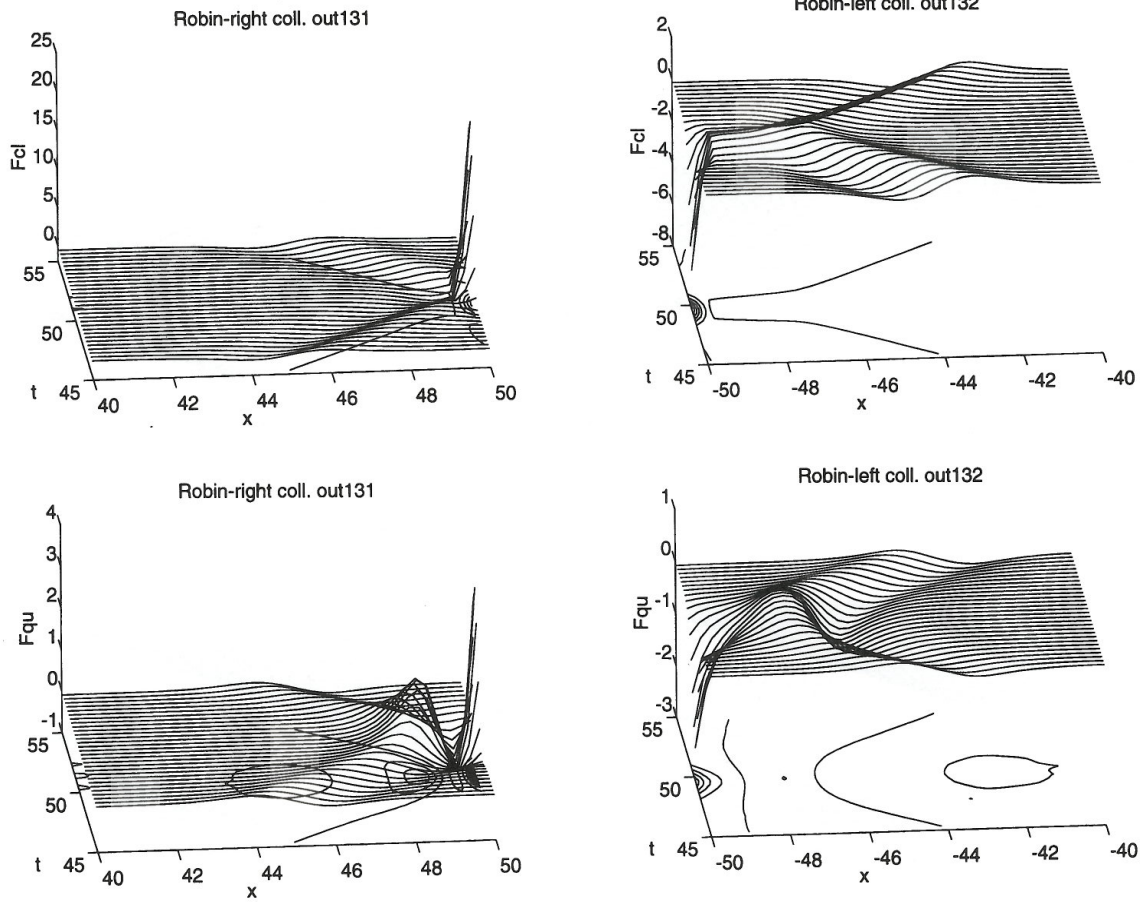
- [6] A. Hasegawa and F. Tappert, “Transmission of Stationary Nonlinear Optical Pulses in Dispersive Dielectric Fibers,” *Appl. Phys. Lett.*, Vol. 23, 1973, pp. 142–144.
- [7] L. D. Landau and E. M. Lifshitz, “*Quantum Mechanics (Non-relativistic Theory)*.” 3rd. ed., Pergamon Press, New York, 1977, p. 13.
- [8] V. E. Zakharov and A. B. Shabat, “Exact Theory of Two-Dimensional Self-Focusing and One-Dimensional Self-Modulation of Waves in Nonlinear Media,” *Sov. Phys. JETP*, Vol. 34, 1972, pp. 62–69.
- [9] J. P. Gordon, “Interaction Forces Among Solitons in Optical Fibers,” *Optics Lett.*, Vol. 8, 1983, pp. 596–598.



**Figure 1** Classical ( $F_{cl}$ ) and quantum ( $F_{qu}$ ) force densities. (Left: infinite waveguide; right: finite waveguide subject to periodic boundary conditions)



**Figure 2** Classical ( $F_{cl}$ ) and quantum ( $F_{qu}$ ) force densities. (Left: finite waveguide subject to homogeneous Dirichlet boundary conditions; right: finite waveguide subject to homogeneous Neumann boundary conditions)



**Figure 3** Classical ( $F_{cl}$ ) and quantum ( $F_{qu}$ ) force densities in a finite waveguide subject to homogeneous Robin boundary conditions. (Left: collision of the soliton with the right boundary; right: collision of the soliton with the left boundary)